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The Natural Structure of Addition

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1. Introduction

At the time being in mathematics addition is assumed to be associative, for example 1+2+3=1+(2+3)=6, while subtraction is not, for example 6-1-2=3 is not equal to 6-(1-2)=7. In this sense the two functions “addition” and “subtraction” seem to have different mathematical structures.
One may now ask, whether this difference in the structures of addition and multiplication exist also in “reality” or the “outside world” (outside our brain) or “nature”, or whether mathematics deviates from “reality” in this respect.
This question can be studied by a physics experiment, inspired by Wittgenstein [1] *:*
somebody goes to the market and buys apples, 1 green apple, 2 red apples, 3 yellow ones, putting them into his basket. This is a very simple counting experiment. It can be recorded by means of a video.
When one evaluates the experiment, one may refer to the basket of the buyer and conclude, that the video shows a process which is associative – apples are added. Instead one may refer to the basket of who sells apples – apples are subtracted -, and one would then conclude that the video shows a process, which is not associative.
So one arrives at two contradicting conclusions, which are however both correct, assuming present day definition of “subtraction”.

2. Addition and subtraction in category theory

The traditional way of adding two numbers, like for instance 1+2=3 can be described in words by saying that “something” with the property “1” and “something” with the property “2” are combined or mapped to “something” with the property “3”. As is known, this very concept, and this very structure can be also expressed in category theory: then addition is considered a coproduct operation, and it has the mathematical structure of a coproduct. In this categoric description “1”, “2” and “3” are objects.
This perspective on addition might derive from the traditional use of the abacus. In the abacus the beads cannot be distinguished from each other, in the sense that they are all equally “objects” to be moved around.

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If instead one expresses what is shown in the video in words, one gets a different description:
First, there is something, which from a physics point of view can be referred to as an object, it is a basket with 1 (for example) apples. Then an action occurs: 1 apple is put into this basket. As a result one obtains a new object, a basket with now 2 apples.
In a categoric description one therefore has a first object “basket with 1 apple”, a morphism “add 1 apple” and a second object “basked with 2 apples”.
So in more general the operation a+b=d takes the form: object(a) -> morphism (+b) -> object(d).
One would consequently define a letter like “a” or “b” to denote an object, while a letter with a symbol, like for instance “+a” or “-a” would denote a corresponding morphism.

In this new scenario it would not make sense to write a+b+c = a+(b+c), since the expression "+(b" would not be defined. Instead, with above convention, we would need to write a+b+c=a(+b+c).
(In this scenario it is not allowed anymore to abbreviate a·(expression) = a(expression).)

now subtraction becomes associative: a-b-c=d would be written as a(-b-c)=d,
for instance 5-2-1=2 = 5(-2-1) = 5-3.

3. Unified algorithm for addition and subtraction

3.1 Addition

Figure 1 shows the algorithm usually used for adding two binary numbers, **a**+**b**=**d**, for the example of figure 3 we assume that **a** and **b** are 3 bits long.

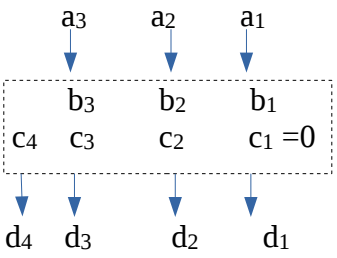


Figure1:algorithm for adding two 3-bit long numbers, **a** and **b**, with **a**+**b**=**d**. “c” is the carry.

The algorithm is written in such a way as to make the categoric nature of the operation visible: an object **a** is mapped by a morphism (shown as dotted line) to an object **d**.
The algorithmic representation of the morphism consists of 3 fundamental adders, each of which
- adds bits a_n and b_n,
- calculates the carry of this addition $c_n = a_{n-1}b_{n-1}+c_{n-1}a_{n-1}+c_{n-1}b_{n-1}$, (I)
- and creates the output $d_n = a_n+b_n+c_n$ (II)

The carry most to the right is always “0”.

3.2 Subtraction

Traditionally, many different algorithms exist for subtraction: computers are using the method of complements, which is fast, but would presumably be very difficult to explain to students, so at schools different methods are used for teaching subtraction, usually making use of borrowing and crutches [2]. Possibly these methods are derived from the use of the abacus.

It is now evident, that the algorithm of figure 1 can be inverted, in which case one obtains **a** from **d** and **b**, which means, a subtraction is now performed, **a**=**d**-**b** :
The execution of the algorithm starts with the adder to the right: since the carry there is always "0" one can get directly $a_1=d_1+b_1$.
With a_1, b_1 and c_1 known, one can calculate c_2, and find $a_2=d_2+b_2+c_2$. Then one continues to the next bit.
Since the algorithm does not distinguish, whether **d**>**b** or **d**<**b**, it calculates only the absolute value of **a**. The correct sign of **a** must be determined from determining separately, which is larger, **b** or **d**. If **d** is larger, **a** must be assigned a negative value.

3.3 unified algorithm for addition and subtraction

Instead of executing one and the same algorithm in one direction for addition, and in the opposite direction for subtraction, one might prefer (for instance for reasons of curiosity) an algorithm which is always executed in the same direction, and which is able to perform both addition or subtraction:

In expression (I) $c_n = a_{n-1}b_{n-1}+c_{n-1}a_{n-1}+c_{n-1}b_{n-1}$, one can with (II) substitute "a_{n-1}" by $a_{n-1}= (d_{n-1}+b_{n-1}+c_{n-1})$
so $c_n = (d_{n-1}+b_{n-1}+c_{n-1})b_{n-1}+c_{n-1}(d_{n-1}+b_{n-1}+c_{n-1})+c_{n-1}b_{n-1}$,
$= b_{n-1}\, d_{n-1}+b_{n-1}+c_{n-1}d_{n-1}+c_{n-1}+c_{n-1}b_{n-1}$,
$= b_{n-1}\, d'_{n-1} + c_{n-1}d'_{n-1} + c_{n-1}b_{n-1}$,
$= b_{n-1}\, d'_{n-1} + c_{n-1}d'_{n-1} + c_{n-1}b_{n-1}$, with d':=(1-d)

This means, that we can calculate the carries either from the components of **a**, as in (I) or instead from **d**'.
We may next denote the carry obtained from **d**' instead of obtained from **a** with "k" instead of "c", this allows us now to create a unified algorithm for addition and subtraction, always executed from top to bottom, as shown in figure 2.

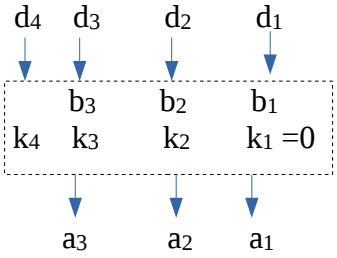


Figure 2: algorithm for subtraction **d**-**b**=**a**.
The carry is now referred to as "k" in order to indicate that it is formed using the inverse of object **d**. The same algorithm can be used for addition, by calculating the carry from **a**.

Similar arguments have been presented for the case of multiplication and division [3].

4. Applications and Outlook

4.1 Teaching mathematics

In the first years at school, children are said to learn “the four basic operations of arithmetic” [3]. Already this terminology is not clear: what is an “operation of arithmetic”? The phrasing suggests, that the operation is somehow mathematics.
Since only the algorithm for addition correctly represents the mathematical structure of this operation, while the algorithms for subtraction, multiplication and division represent complicated antique mnemonic procedures, school children learn algorithms, but believe to learn mathematics.
As a result, many children suffer and later as adults wrongly believe not to understand mathematics and even to hate mathematics.
Teaching the much simpler algorithms which do have the correct mathematical properties, would allow children not only to learn arithmetics faster and easier, but it would also allow them to see the beauty and simplicity of mathematics. They would learn to know mathematics as a friend, not an enemy.

4.2 Mathematics and reality

Nowadays, the relation between physics and mathematics, and therefore “nature” or “reality” and mathematics may be summarized in the famous Einstein quote “Insofar as mathematical propositions refer to reality, they are not certain, and insofar as they are certain, they do not refer to reality“, which may be a little bit exaggerated, but is not utterly wrong.
The examples of division and subtraction indicate, that discrepancies between mathematics and reality might only be due to the use of the “wrong” algorithms, where “wrong” means an algorithm, which is able to perform a correct mapping of input to output, but without having the structure of the respective mathematical function.

5. Conclusion

The algorithms used nowadays for subtraction are not associative. This seems to create a difference between mathematics and physics, since in physics experiments subtraction – being the inverse of addition – should be associative.
Based on a categorical approach, it is possible to create an algorithm which can be applied for addition as well as for subtraction, and which gives an associative structure also to subtraction.
1) Teaching physics at school in this way should make the first years of mathematical education easier for students.
2) The findings of this article confirm what was suggested in a recent article on multiplication and division: procedures of information processing (here the operation of subtraction) may very well perform a correct mapping of input to output, but in spite of this they may not have the mathematical structure of the function, they perform. Considering instead the mathematical structure of the mapping leads to a better agreement between the world of mathematics and the physical world.

References

[1] Ludwig Wittgenstein: *Philosophische Untersuchungen*, Suhrkamp Verlag, 9. Auflage 2003, https://www.wittgensteinproject.org/w/index.php/Philosophische_Untersuchungen

[2] https://en.wikipedia.org/wiki/Subtraction

[3] A new method for division and factorization based on inverse computing Prolegomena to a physics of information), H.Grassmann, Isomorph Letters A, Physics of Information, 1 (2025) https://isomorph-letters.it/

[4] https://en.wikipedia.org/wiki/Addition