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Manuscript: "The Natural Structure of Addition" by Hans Grassmann, Daria Botsula, Parsu Ram Sharma

1. By definition, a binary mathematical operation $*$ is associative if changing the grouping of the terms gives the same answer, that is, $(x * y) * z = x * (y * z)$. It is easy to verify that addition is associative, but subtraction is not. The authors of this letter ask: "Why is this so?" And what about objective reality, independent of our thinking?

Wittgenstein's physical experiment, in which the buying and selling of apples are viewed as the same operation from opposite sides, demonstrates the contradictory nature of conclusions regarding the associativity of addition and the non-associativity of subtraction.

2. The authors analyze this problem using category theory methods.

They note that the traditional way of adding numbers can actually be interpreted from the position of category theory as the mathematical structure of the coproduct (that are alternatively known as coprojection) of abstract objects. This idea is suggested by the use of a regular abacus. From the point of view of physics, there are objects and actions with them. Then in the abstract sense the operation $a + b = d$ has the form: object(a) -> morphism ($+b$) -> object(d), and morphism ($-b$) would denote subtraction. With this convention, we would need to write $a + b + c = a(+b + c)$ and now subtraction becomes associative: $a - b - c = d$ have to be written as $a(-b - c) = d$.

I note that the authors are right in trying to find a more convenient form of notation for addition and subtraction in order to avoid asymmetry in the associativity property. They achieve this by combining the object and the morphism in the notation, that is, by putting a number with a plus or minus sign in brackets. It is a fine idea. But it is unlikely that the notations they proposed are more perfect than the modern ones. Why? At first, the expression of type $a(-b - c)$ one may identify with the product a with $(-b - c)$, and secondly, for right thinking each morphism should be presented explicitly. It would be another matter if the authors proposed to introduce one single notation for the morphism of addition and subtraction, and the numbers were immediately denoted with the corresponding sign, depending on whether it is positive or negative.

3. Universal algorithm for addition and subtraction.

Here authors analyze the usually used algorithm for adding two binary numbers using schematic figure. The unified addition and subtraction algorithm proposed by the authors allows for both addition and subtraction to be performed using a single algorithm in one direction.

The algorithm makes the categorial nature of the operation visible: an object a is mapped by a morphism adding to an object d. The algorithmic representation of the morphism consists of 3 fundamental adders. In fact authors proposed to change the notation of this operations. They argue in quite detail and correctly. After a fairly in-depth

analysis of traditional algorithms, the authors present their version of a universal algorithm for addition and subtraction operations. Of course, this is the grand achievement of this note: to propose a notation for addition and subtraction that ensures the associativity property of these operations.

It would be appropriate for the authors to demonstrate their universal algorithm for finding the result in an example with positive and negative numbers, especially when you need to not only add and subtract, but also multiply and divide. Then the extra parentheses could lead to confusion.

I repeat the same as above remark applies even more to the content of this place in the article where the authors create the unified algorithm for addition and subtraction. Using brackets without explicit notation for morphism may leads to confusions.

It is worth recalling that in higher mathematics, the symbol Σ is typically used for the addition of both positive and negative numbers, although a different symbol Δ is often employed to denote a difference. In continuous case also used two opposite symbols $\int$ and derivation $\frac{d}{dx}$, analogous to summation and differentiation. In general, the introduction of a new symbol in mathematics is a highly innovative step.

4. The last part of the paper is higher interesting.

Teaching Mathematics. The authors are right that only the addition algorithm correctly reflects the mathematical structure of this operation, and that schoolchildren learn algorithms, but not mathematics. Thus, teaching simpler algorithms may allow children to learn arithmetic more quickly and easily, and in addition, for some of them to see the beauty of real mathematics.

Mathematics and reality. Here the authors touch on a very delicate topic: the connection of physics with mathematics, or rather the real world with ideal (mathematical concepts are ideal, they do not exist in nature, but only in consciousness).

Many prominent thinkers have pondered the question of which of these spheres the propositions are true. Practical algorithm or mathematical theorem? An open discussion of this topic is extremely relevant, especially in the modern era of rapid development of AI.

I strongly support this kind of publication, as they encourage an open discussion about the role and methods of studying mathematics in the process of forming thinking in schoolchildren. What is more correct, algorithms of pressing computer buttons or attempts to understand the simplest logic of mathematical problems? This question is extremely relevant in our time, when the IQ level of AI can exceed the IQ level of most people.